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Inverse problem of imposing nodes to suppress vibration for a structure subjected to multiple harmonic excitations

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Abstract

Undamped oscillators are frequently used as a means to minimize excess vibration in flexible structures. In this paper, parallel vibration absorbers are employed to solve the inverse problem of dictating a node location, i.e., a point of zero vibration, for an arbitrarily supported linear structure subjected to multiple harmonic excitations. When the attachment location for the oscillators and the desired node location coincide (or collocated), it is always possible to select a set of spring–mass parameters such that a node is induced at the desired location for an input consisting of multiple harmonics. When the attachment and the specified node locations are not collocated, however, it is only possible to induce a node at certain locations along the elastic structure for the same input. When the input consists of two harmonics with closely spaced frequencies, it is possible to induce a point of nearly zero amplitude for frequencies in the range between the two driving frequencies. Moreover, when the specified node locations are in the vicinity of one another, a region of nearly zero deflections can be enforced, effectively quenching vibration in that segment of the structure. Finally, the proposed algorithm can be easily modified and extended to enforcing more than one node when the structure is being excited by multiple harmonics, for both the collocated and non-collocated cases. A procedure to guide the proper selection of the spring–mass parameters is outlined in detail, and numerical case studies are presented to verify the utility of the proposed scheme of imposing

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one or multiple nodes for an arbitrarily supported linear structure subjected to an input consisting of multiple harmonics.

1. Introduction

Over the years a lot of work has been done on using vibration absorbers to control and to minimize excess vibration and sound radiation in structural systems, hence only a few selected references are cited here [1–13]. Interestingly, properly tuned spring–mass systems can also be used to dictate the node locations or points of zero displacement, thereby offering an alternative means of quenching excess vibration.

In Ref. [14], Cha and Pierre used a chain of oscillators to passively impose a single node for the normal modes of any arbitrarily supported elastic structure. The desired node can either coincide with the oscillator chain or it can be located elsewhere. A procedure to guide the proper selection of the oscillator chain parameters for the purpose of inducing a single node for multiple normal modes was outlined in detail. In Ref. [15], Cha developed an approach that used parallel sprung masses to induce multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. By selecting the appropriate sprung masses, their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing the locations of the nodes to be specified anywhere along the structure and for any normal mode. The focus of Refs. [14,15] was on imposing nodes for the normal modes. In Ref. [16], spring–mass systems were used to induce a single or multiple nodes anywhere along an elastic structure that is harmonically excited with a localized force, subjected to the constraints of tolerable vibration amplitude of the masses. An efficient procedure for tuning the sprung masses was proposed, and numerical experiments validated the utility of the approach. While useful, the methodology applies only when the structure is subjected to a single harmonic. Unfortunately, physical inputs seldom consist of a single harmonic excitation.

In this paper, elastically mounted masses are used to induce a single or multiple nodes anywhere along an arbitrarily supported structure that is under general dynamic loads. The proposed scheme can be used to suppress excess vibration for a structure that is subjected to any arbitrary input, as long as its spectrum and dominant frequencies are known. The ability to impose a single or multiple nodes anywhere on the structure is beneficial because it offers an alternative approach to quenching excess vibration. Moreover, it allows delicate and sensitive instruments to be mounted near or at a region where there is little or no vibration. Thus, the proposed scheme allows certain points along the structure to remain stationary without using any rigid supports. Interestingly, the same solution scheme can be easily extended to accommodate other cases. For example, when the input consists of two harmonics with closely spaced frequencies, it is possible to induce a point of nearly zero amplitude for frequencies in the range between the two driving frequencies. This enables any point along the structure to remain nearly motionless when the structure is driven by a single harmonic whose excitation frequency has a tendency to drift. Finally, the algorithm can also be applied to impose multiple nodes anywhere along the structure. This is beneficial because if the nodes are closely spaced, a region of nearly zero amplitudes can be enforced, thereby quenching vibration in that segment of the structure when it is subjected to an input with multiple harmonics.

2. Theory

Consider an arbitrarily supported linear structure carrying a set of S parallel oscillators as shown in Fig. 1. A localized force

$$f(t) = \sum_{i=1}^{p} F_i \mathrm{e}^{\mathrm{j}\omega_i t} \tag{1}$$

is applied to the structure at x_f, where p is the number of harmonic excitations, F_i and ω_i represent the forcing amplitude and forcing frequency of the ith harmonic, respectively, and j denotes the imaginary unit. Using the assumed-modes method [17], the physical deflection of the structure at a point x is simply

$$w(x,t) = \sum_{i=1}^{N} u_i(x)\eta_i(t), \tag{2}$$

where $u_i(x)$ are the eigenfunctions of the unconstrained structure (i.e., the structure without any attachments) that serve as the basis functions for this approximate solution, $\eta_i(t)$ are the corresponding generalized coordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic energy of the combined system is

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} m_i\dot{z}_i^2(t), \tag{3}$$

where M_i are the generalized masses, m_i is the mass of the ith oscillator, $z_i(t)$ is its displacement, S is the total number of oscillators that are attached to the structure, and an overdot denotes a derivative with respect to time. The total potential energy is

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} k_i[z_i(t) - w(x_a,t)]^2, \tag{4}$$

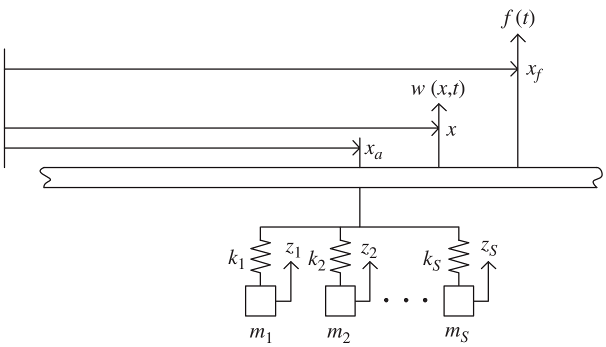


Fig. 1. An arbitrarily supported elastic structure that is subjected to a localized force and carrying a set of sprung masses.

where K_i are the generalized spring constants, k_i is the spring stiffness of the ith oscillator, x_a represents the attachment location of the oscillators, and $w(x_a, t)$ represents the lateral displacement of the beam at x_a.

Applying Lagrange's equations, the equations of motion for the system are given by

$$\begin{bmatrix} [\mathscr{M}] & [0] \\ [0]^{\mathrm{T}} & [m] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \ddot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} [\mathscr{K}] & [R] \\ [R]^{\mathrm{T}} & [k] \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} f(t)\mathbf{u}(x_f) \\ \mathbf{0} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{\eta} = [\eta_1 \quad \eta_2 \ldots \eta_N]^{\mathrm{T}}$ is the vector of generalized coordinates and $\mathbf{z} = [z_1 \quad z_2 \ldots z_S]^{\mathrm{T}}$ is the vector of mass displacements. The $S \times S$ matrices $[m]$ and $[k]$ are

$$[m] = \mathrm{diag}[m_i], [k] = \mathrm{diag}[k_i]. \tag{6}$$

The $N \times N$ matrices $[\mathscr{M}]$ and $[\mathscr{K}]$ are

$$[\mathscr{M}] = [M^d], [\mathscr{K}] = [K^d] + \sum_{i=1}^{S} k_i \mathbf{u}(x_a)\mathbf{u}^{\mathrm{T}}(x_a), \tag{7}$$

where $[M^d]$ and $[K^d]$ are both diagonal matrices whose ith elements are M_i and K_i, respectively. Vector $\mathbf{u}(x_a)$ is defined as

$$\mathbf{u}(x_a) = [u_1(x_a) \quad u_2(x_a) \ldots u_N(x_a)]^{\mathrm{T}}, \tag{8}$$

and the $N \times S$ matrices $[R]$ and $[0]$ are given by

$$[R] = [-k_1\mathbf{u}(x_a) \ \ldots \ -k_i\mathbf{u}(x_a) \ \ldots \ -k_S\mathbf{u}(x_a)], [0] = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \ldots \quad \mathbf{0}]. \tag{9}$$

Because the system is linear, superposition can be applied. Thus, the total steady-state response of the system to a given number of distinct harmonic excitations can be obtained separately and then combined to obtain the aggregate response. To this end, one can consider a general harmonic input of the form

$$f(t) = F\mathrm{e}^{\mathrm{j}\omega t}. \tag{10}$$

The system will execute a simple harmonic motion with the same response frequency as the driving frequency,

$$\eta_i(t) = \bar{\eta}_i \,\mathrm{e}^{\mathrm{j}\omega t}, z_i(t) = \bar{z}_i \,\mathrm{e}^{\mathrm{j}\omega t}. \tag{11}$$

Therefore, vectors $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1 \quad \bar{\eta}_2 \ldots \bar{\eta}_N]^{\mathrm{T}}$ and $\bar{\mathbf{z}} = [\bar{z}_1 \quad \bar{z}_2 \ldots \bar{z}_S]^{\mathrm{T}}$ correspond to the solution of the following matrix equation

$$\begin{bmatrix} [\mathscr{K}] - \omega^2[\mathscr{M}] & [R] \\ [R]^{\mathrm{T}} & [k] - \omega^2[m] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} F\mathbf{u}(x_f) \\ \mathbf{0} \end{bmatrix}. \tag{12}$$

Incidentally, by setting the right-hand side of Eq. (12) to zero, one obtains

$$\begin{bmatrix} [\mathscr{K}] & [R] \\ [R]^{\mathrm{T}} & [k] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & [0] \\ [0]^{\mathrm{T}} & [m] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix}, \tag{13}$$

which corresponds to the generalized eigenvalue problem for a linear structure carrying S oscillators. In Eq. (13), ω denotes the natural frequency of the combined system.

While Eq. (12) is expressed in terms of the generalized coordinates $\bar{\boldsymbol{\eta}}$ and the mass amplitudes $\bar{\mathbf{z}}$, it can be easily manipulated to depend only on $\bar{\boldsymbol{\eta}}$. The last S equations of Eq. (12) yield

$$-k_i\mathbf{u}^{\mathrm{T}}(x_a)\bar{\boldsymbol{\eta}} + (k_i - \omega^2 m_i)\bar{z}_i = 0\,,\ i = 1,\ldots,S. \tag{14}$$

Eq. (12) also gives

$$\big([\mathscr{K}] - \omega^2[\mathscr{M}]\big)\bar{\boldsymbol{\eta}} - \sum_{i=1}^{S} k_i\mathbf{u}(x_a)\bar{z}_i = F\mathbf{u}(x_f). \tag{15}$$

Solving for $\bar{z}_i$ by using Eq. (14) and substituting the resulting expression into Eq. (15) leads to

$$\big([K^d] + \alpha\mathbf{u}(x_a)\mathbf{u}^{\mathrm{T}}(x_a) - \omega^2[M^d]\big)\bar{\boldsymbol{\eta}} = F\mathbf{u}(x_f), \tag{16}$$

where

$$\alpha = \sum_{i=1}^{S}\frac{k_i m_i\omega^2}{m_i\omega^2 - k_i}. \tag{17}$$

Note that the coefficient matrix of $\bar{\boldsymbol{\eta}}$ is simply the sum of a diagonal matrix and a rank one matrix.

Assuming that the excitation frequency does not coincide with any natural frequencies of the modified system, i.e., the linear structure carrying the chain of oscillators, the coefficient matrix of Eq. (16) can be inverted to give

$$\bar{\boldsymbol{\eta}} = \big([K^d] + \alpha\mathbf{u}(x_a)\mathbf{u}^{\mathrm{T}}(x_a) - \omega^2[M^d]\big)^{-1}F\mathbf{u}(x_f). \tag{18}$$

To induce a node at x_n requires that

$$w(x_n,t) = \sum_{i=1}^{N} u_i(x_n)\eta_i(t) = \mathbf{u}^{\mathrm{T}}(x_n)\boldsymbol{\eta} = \mathbf{u}^{\mathrm{T}}(x_n)\bar{\boldsymbol{\eta}}\,\mathrm{e}^{\mathrm{j}\omega t} = 0. \tag{19}$$

Combining Eqs. (18) and (19), one gets

$$\mathbf{u}^{\mathrm{T}}(x_n)\big([K^d] + \alpha\mathbf{u}(x_a)\mathbf{u}^{\mathrm{T}}(x_a) - \omega^2[M^d]\big)^{-1}F\mathbf{u}(x_f) = 0. \tag{20}$$

Because the second term of Eq. (20) consists of a diagonal matrix modified by a rank one matrix, its inverse can be readily obtained by applying the Sherman–Morrison formula [18]. Expanding the triple product of Eq. (20) yields

$$c_1 - \frac{\alpha}{1 + c_3\alpha}\,c_2 = 0\,, \tag{21}$$

where

$$c_1 = \sum_{i=1}^{N}\frac{u_i(x_n)u_i(x_f)}{K_i - M_i\omega^2}\,, \tag{22}$$

$$c_2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{u_i(x_a)u_j(x_a)u_i(x_n)u_j(x_f)}{(K_i - M_i\omega^2)(K_j - M_j\omega^2)}\,, \tag{23}$$

and

$$c_3 = \sum_{i=1}^{N} \frac{u_i^2(x_a)}{K_i - M_i\omega^2}. \tag{24}$$

Eqs. (20) and (21) are mathematically identical. Eq. (21), however, is clearly more efficient, because it does not require one to perform the computationally taxing operation of inverting an $N \times N$ matrix.

The required oscillator parameters to impose a node when the structure is driven by multiple harmonics are obtained numerically. Eq. (21) depends on x_n, x_a, x_f, m_i, k_i and ω. It can be conveniently expressed in terms of these dependent variables as

$$f(x_a, x_f, x_n, \mathbf{m}, \mathbf{k}, \omega) = 0, \tag{25}$$

where $\mathbf{m}$ and $\mathbf{k}$ are the vector of oscillator masses and stiffnesses, respectively. In application, it is always possible to attach more oscillators than there are excitation frequencies, i.e., $S > p$. However, it is clearly more efficient to use the fewest number of sprung masses possible. Thus, for p excitation frequencies, $S = p$ oscillators are attached at x_a of the linear structure to induce a node at x_n, and Eq. (25) leads to a set of p equations, one for each ω_i, of the following form

$$f_i(x_a, x_f, x_n, \mathbf{m}, \mathbf{k}, \omega_i) = 0, \; i = 1, \ldots, p. \tag{26}$$

Assuming that the stiffness parameters k_i are all specified, Eq. (26) yields a set of p nonlinear algebraic equations that must be solved simultaneously for the p masses m_i.

The MATLAB routine *fsolve* is utilized in this paper to obtain the solution of a system of nonlinear algebraic equations using a quasi-Newton method. To execute *fsolve*, a set of initial guesses must be provided for the unknowns. For these initial guesses, if *fsolve* does not converge to a solution, then *fsolve* is executed again with a different set of starting values until a solution is obtained. Mathematically, if the set of equations given by Eq. (26) returns any mass value that is negative, then a node cannot be enforced at the desired location for the given set of ω_i, k_i, x_a and x_f. In this case, one has the option to change either the oscillator stiffnesses, k_i, or the attachment location, x_a, to obtain physically meaningful, i.e., positive, values of m_i so that a node at x_n can be induced for the given x_f and ω_i. The proposed technique of solving for the masses in order to impose a node at x_n is very robust. In all of the numerical experiments considered, *fsolve* successfully converged to a set of theoretically feasible solutions.

For multiple harmonics, the input is given by Eq. (1). Because the structure under consideration is assumed to be linear, superposition is valid and one can consider each input individually. For each harmonic excitation with frequency, ω_r, the deformed shape of the structure can be expressed as

$$w_r(x,t) = \sum_{i=1}^{N} u_i(x)\eta_i^r(t) = \phi_r(x)e^{j\omega_r t}, \tag{27}$$

where $\eta_i^r(t)$ denotes the generalized coordinates due to a harmonic input of $F_r e^{j\omega_r t}$, and $\phi_r(x)$ represents the resulting deflection shape of the beam. Thus, by virtual of superposition, the total response of the structure is given by

$$w(x,t) = \sum_{r=1}^{p} \phi_r(x)e^{j\omega_r t}. \tag{28}$$

3. Results

In the subsequent examples, all of the oscillator stiffnesses will be specified, and all of the excitation frequencies are arbitrary but distinct from the natural frequencies of the combined system, which consists of the elastic structure carrying the set of oscillators. Moreover, to ensure the convergence for all of the numerical results, the number of modes used in the expansion is taken to be $N = 10$. Finally, because the assumed-modes method is used to formulate the equations of motion, the proposed procedures can be easily implemented to enforce a node for any arbitrarily supported structure undergoing multiple harmonic excitations. Without any loss of generality, a simply supported and a fixed-free uniform Euler–Bernoulli beam will be considered.

For a uniform simply supported Euler–Bernoulli beam, its normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions are given by

$$u_i(x) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{i\pi x}{L}\right), \tag{29}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \text{ and } K_i = (i\pi)^4 EI/(\rho L^4), \tag{30}$$

where E is the Young's modulus, I is the moment of inertia of the cross-section of the beam. For a uniform fixed-free Euler–Bernoulli beam, its normalized eigenfunctions are

$$u_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L}(\sin \beta_i x - \sinh \beta_i x)\right), \tag{31}$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \text{ and } K_i = (\beta_i L)^4 EI/(\rho L^4), \tag{32}$$

Table 1
The first five natural frequencies of a uniform simply supported and a uniform fixed-free beam

Natural frequency	Simply supported	Fixed-free
ω'_1	9.86960E+00	3.51602E+00
ω'_2	3.94784E+01	2.20345E+01
ω'_3	8.88264E+01	6.16972E+01
ω'_4	1.57914E+02	1.20902E+02
ω'_5	2.46740E+02	1.99860E+02

The natural frequencies, ω'_i, of the combined system are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

where $\beta_i L$ satisfies the following transcendental equation

$$\cos \beta_i L \cosh \beta_i L = -1. \quad (33)$$

Table 1 lists the first five natural frequencies of a uniform simply supported and a uniform fixed-free Euler–Bernoulli beam.

Two cases will be considered in detail. In the first case, the attachment and the node locations coincide. This will be referred to as the collocated case. In the second case, the attachment and the node locations are distinct, otherwise known as the non-collocated case. In application, the procedure outlined in the previous section can be used to determine the mass parameters in order to induce a node at x_n, regardless if the attachment and node locations are collocated or non-collocated. However, as will be shown shortly, when the attachment and the node locations are collocated, then the procedure to find the required oscillator parameters becomes trivial.

Eq. (26) will be used to find the required masses in order to impose a node at x_n for multiple harmonic excitations. To validate the results of the assumed modes method and the mass parameters of Eq. (26), a finite element model of the beam carrying sprung masses is constructed, where the stiffnesses of the oscillators correspond the specified values, and the masses of the oscillators are given by the solutions of Eq. (26). The finite element model will also be subjected to the same localized input consisting of multiple harmonics. In all of the subsequent examples, the deformed shapes of the beam for each harmonic excitation, the natural frequencies of the structure carrying oscillators, and the mass amplitudes will be determined using both the assumed modes and the finite element method. In using the latter approach, the beam is discretized into 100 finite elements of equal length.

3.1. Collocated case: one node

Consider the case where the attachment and the node locations coincide, $x_a = x_n$. For this collocated case, if

$$k_i = m_i\omega_i^2 \quad i = 1, \ldots, p, \quad (34)$$

then Eq. (14) reduces to

$$-k_i\mathbf{u}^{\mathrm{T}}(x_a)\bar{\boldsymbol{\eta}} = -k_i\mathbf{u}^{\mathrm{T}}(x_n)\bar{\boldsymbol{\eta}} = 0, \quad i = 1, \ldots, p, \quad (35)$$

which clearly satisfies Eq. (19). Thus, to induce a node at the attachment location, the natural frequencies of the grounded oscillators must be identical to the excitation frequencies. If the stiffnesses of the oscillators are specified, the required masses are found directly from Eq. (34). Once the oscillator parameters are properly tuned based on the excitation frequencies, the attachment location of the oscillators, regardless of its position along the structure, becomes a node. Incidentally, the selection of the sprung masses is not unique. The actual choice is governed by the tolerable vibration amplitudes of the oscillator masses.

Attaching oscillators to any linear structure changes the natural frequencies of the system. Because the oscillator parameters are tuned so that the grounded natural frequency of each oscillator matches one of the excitation frequencies exactly (see Eq. (34)), some of the natural frequencies of the combined assembly (the beam carrying the oscillator attachments) may be close, but will never be identical, to the driving frequencies. This, however, generally presents no problem because the new natural frequencies differ from the operating frequencies.

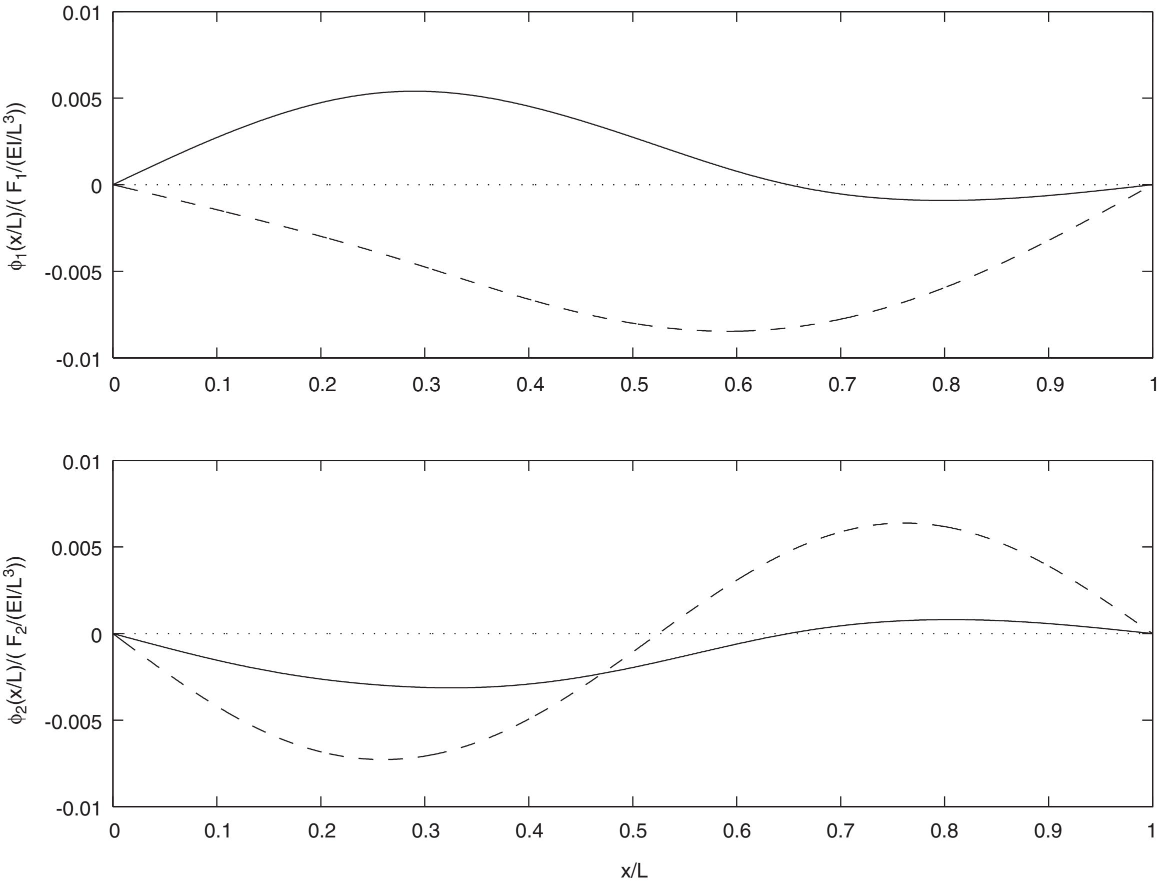


Fig. 2. The normalized steady-state deformed shape of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The horizontal line represents the configuration of the undeformed beam. The system parameters are $k_1 = k_2 = 5EI/L^3$, $m_1 = 1.73010 \times 10^{-2}\rho L$ and $m_2 = 2.70416 \times 10^{-3}\rho L$. The attachment and node locations are collocated, $x_a = x_n = 0.65L$. The excitation frequencies are $\omega_1 = 17\sqrt{EI/(\rho L^4)}$, $\omega_2 = 43\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.27L$.

Consider a uniform simply supported Euler–Bernoulli beam of length L, subjected to a localized force at $x_f = 0.27L$ of the form

$$f(t) = F_1 e^{j\omega_1 t} + F_2 e^{j\omega_2 t}, \tag{36}$$

where $\omega_1 = 17\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 43\sqrt{EI/(\rho L^4)}$. Comparing with the results of Table 1, note that the excitation frequencies are distinct from the natural frequencies of the simply supported beam. For a given application, the attachment and the desired node locations are collocated, i.e., $x_a = x_n = 0.65L$. Two oscillators are attached to the beam at x_a, and the stiffness parameters are chosen to be $k_1 = k_2 = 5EI/L^3$. To induce a node at the attachment location, one simply requires $m_i = k_i/\omega_i^2$, or $m_1 = 1.73010 \times 10^{-2}\rho L$ and $m_2 = 2.70416 \times 10^{-3}\rho L$. Fig. 2 shows the normalized steady-state deformed shape of the beam, $\phi_r(x/L)/(F_r/(EI/L^3))$, due to $F_r e^{j\omega_r t}$. The solid curve corresponds to the deformed shape of the beam with two oscillators attached at $x_a = 0.65L$. The dotted line corresponds to the deformed shape of the beam without the oscillators, and the horizontal line represents the undeformed configuration of the beam. The total steady-state response of the beam is given by Eq. (28). For a single harmonic excitation, the amplitude of the beam's deformation changes but the beam's deformed shape remains the same as time evolves. For an input consisting of multiple harmonics with distinct excitation frequencies, the resulting amplitude and the deformed shape both vary with time. Nevertheless, by attaching oscillators with properly chosen system parameters, its attachment location, $x_a = 0.65L$, can be made to always remain stationary. Note also that by attaching two oscillators of the appropriate parameters at $x_a = 0.65L$, the beam in the region between $0.65L \leqslant x \leqslant 1.00L$ experiences much smaller vibration compared to the beam without sprung masses.

To validate the results, a finite element model of the beam carrying two oscillators is developed. The deformed shapes of the finite element beam model are identical to those of Fig. 2 and they will not be shown. Table 2 compares the natural frequencies of the combined system (the beam carrying the two oscillators), obtained by using the assumed modes and the finite element method. Note the excellent agreement between the two approaches. From Table 2, observe that the excitation frequencies are now near the second and fourth natural frequencies of the combined system.

Table 3 compares the mass amplitudes for each harmonic excitation. Note again how well they track one another. Because the first oscillator is tuned to ω_1, it alone can be used to induce a node at the attachment location when the beam is being excited by a single harmonic with frequency ω_1. Thus, when the input consists of $F_1 e^{j\omega_1 t}$, the second oscillator remains completely stationary, and the results of Table 3 are consistent with our physical understanding of the problem. Incidentally, in all of the subsequent examples, the assumed modes and the finite element results agree very well. Thus, unless otherwise stated, the deformed shapes, natural frequencies and mass amplitudes obtained by using the finite element method will not be shown for the sake of brevity.

Because the mass parameters are obtained by using $m_i = k_i/\omega_i^2$, they are certainly not unique. In application, another important design specification is governed by the vibration of the absorber masses. For the previous example, if the mass amplitudes are deemed too large, one can simply

Table 2
The first six natural frequencies of a uniform simply supported Euler-Bernoulli beam carrying two undamped oscillators, of stiffnesses $k_1 = k_2 = 5EI/L^3$ and masses $m_1 = 1.73010 \times 10^{-2}\rho L$ and $m_2 = 2.70416 \times 10^{-3}\rho L$, at $x_a = 0.65L$

Natural frequency	Assumed modes ($N = 10$)	Finite element ($N = 100$)
ω_1'	9.65390E+00	9.65390E+00
ω_2'	1.72914E+01	1.72913E+01
ω_3'	3.91795E+01	3.91795E+01
ω_4'	4.35249E+01	4.35246E+01
ω_5'	8.88297E+01	8.88297E+01
ω_6'	1.57974E+02	1.57974E+02

The natural frequencies, ω_i', of the combined system are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 3
The mass amplitudes for the beam and oscillators of Table 2

Mass amplitudes	Assumed modes ($N = 10$)	Finite element ($N = 100$)
z_1 (for ω_1)	−2.31255E−01	−2.31342E−01
z_2 (for ω_1)	−5.92553E−14	−5.92763E−14
z_1 (for ω_2)	4.50650E−14	4.50848E−14
z_2 (for ω_2)	1.78868E−01	1.78948E–01

The beam is subjected to a localized force at $x_f = 0.27L$ that consists of two harmonics with frequencies $\omega_1 = 17\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 43\sqrt{EI/(\rho L^4)}$. The mass amplitudes are normalized by dividing by $F_r/(EI/L^3)$, where F_r denotes the forcing amplitude of the rth harmonic.

Table 4
The first six natural frequencies of a uniform simply supported Euler-Bernoulli beam carrying two undamped oscillators, of stiffnesses $k_1 = k_2 = 20EI/L^3$ and masses $m_1 = 6.92041 \times 10^{-2}\rho L$ and $m_2 = 1.08167 \times 10^{-2}\rho L$, at $x_a = 0.65L$

Natural frequency	Assumed modes ($N = 10$)	Finite element ($N = 100$)
ω'_1	9.11010E+00	9.11009E+00
ω'_2	1.80487E+01	1.80482E+01
ω'_3	3.85681E+01	3.85680E+01
ω'_4	4.48111E+01	4.48100E+01
ω'_5	8.88395E+01	8.88395E+01
ω'_6	1.58154E+02	1.58154E+02

The natural frequencies, ω'_i, of the combined system are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 5
The mass amplitudes for the beam and oscillators of Table 4

Mass amplitudes	Assumed modes ($N = 10$)	Finite element ($N = 100$)
z_1 (for ω_1)	−5.78136E–02	−5.78355E–02
z_2 (for ω_1)	−4.90571E–15	−4.90694E–15
z_1 (for ω_2)	3.60698E–15	3.60824E–15
z_2 (for ω_2)	4.47170E−02	4.47369E−02

The beam is subjected to a localized force at $x_f = 0.27L$ that consists of two harmonics with frequencies $\omega_1 = 17\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 43\sqrt{EI/(\rho L^4)}$. The mass amplitudes are normalized by dividing by $F_r/(EI/L^3)$, where F_r denotes the forcing amplitude of the rth harmonic.

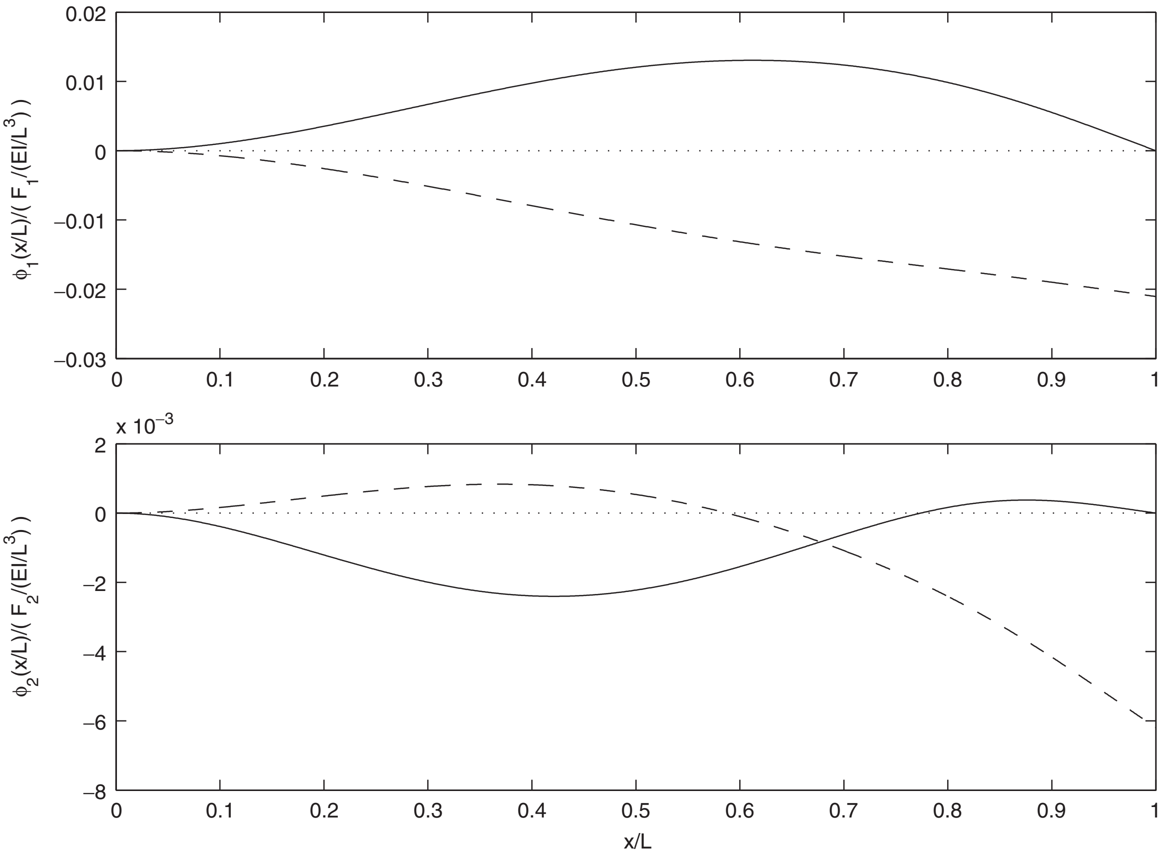


Fig. 3. The normalized steady-state deformed shape of a uniform fixed-free Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $k_1 = k_2 = 7EI/L^3$, $m_1 = 4.86111 \times 10^{-2}\rho L$ and $m_2 = 7.28408 \times 10^{-3}\rho L$. The attachment and node locations are collocated, $x_a = x_n = 1.00L$. The excitation frequencies are $\omega_1 = 12\sqrt{EI/(\rho L^4)}$, $\omega_2 = 31\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.85L$.

increase the spring stiffnesses to lower these amplitudes to within an acceptable level. Changing the spring stiffnesses will not alter the deformed shape of the beam, as long as they satisfy $m_i = k_i/\omega_i^2$. They will, however, change the natural frequencies of the combined system and the amplitudes of the sprung masses. Table 4 and Table 5 show the natural frequencies and mass amplitudes for the case of $k_1 = k_2 = 20EI/L^3$. As expected, by increasing the stiffness parameters, the absorber amplitudes decrease. Note that the excitation frequencies are in the vicinity of the second and fourth natural frequencies of the combined assembly.

Consider a fixed-free beam that is subjected to a localized input at $x_f = 0.85L$, which consists of two harmonics with frequencies $\omega_1 = 12\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 31\sqrt{EI/(\rho L^4)}$. A node is desired at $x_n = 1.00L$, and it coincides with the attachment location. For $k_1 = k_2 = 7EI/L^3$, the required mass parameters are $m_1 = 4.86111 \times 10^{-2}\rho L$ and $m_2 = 7.28408 \times 10^{-3}\rho L$. Fig. 3 shows the normalized steady-state deformed shape of the beam, $\phi_r(x/L)/(F_r/(EI/L^3))$, due to $F_r e^{j\omega_r t}$. Without any attachments, the maximum displacement of the beam occurs at its free end. By attaching properly tuned oscillators to the beam, even though the beam is cantilevered, a node is induced at the beam's tip without using any rigid supports. Interestingly, the deformed shapes of the fixed-free beam with attachments resemble those for a fixed-simply supported beam.

Consider a beam that is subjected to an input with two harmonics that are closely spaced, i.e., $\omega_1 \approx \omega_2$. If a node is induced at x_n for ω_1 and ω_2, that particular location may remain nearly

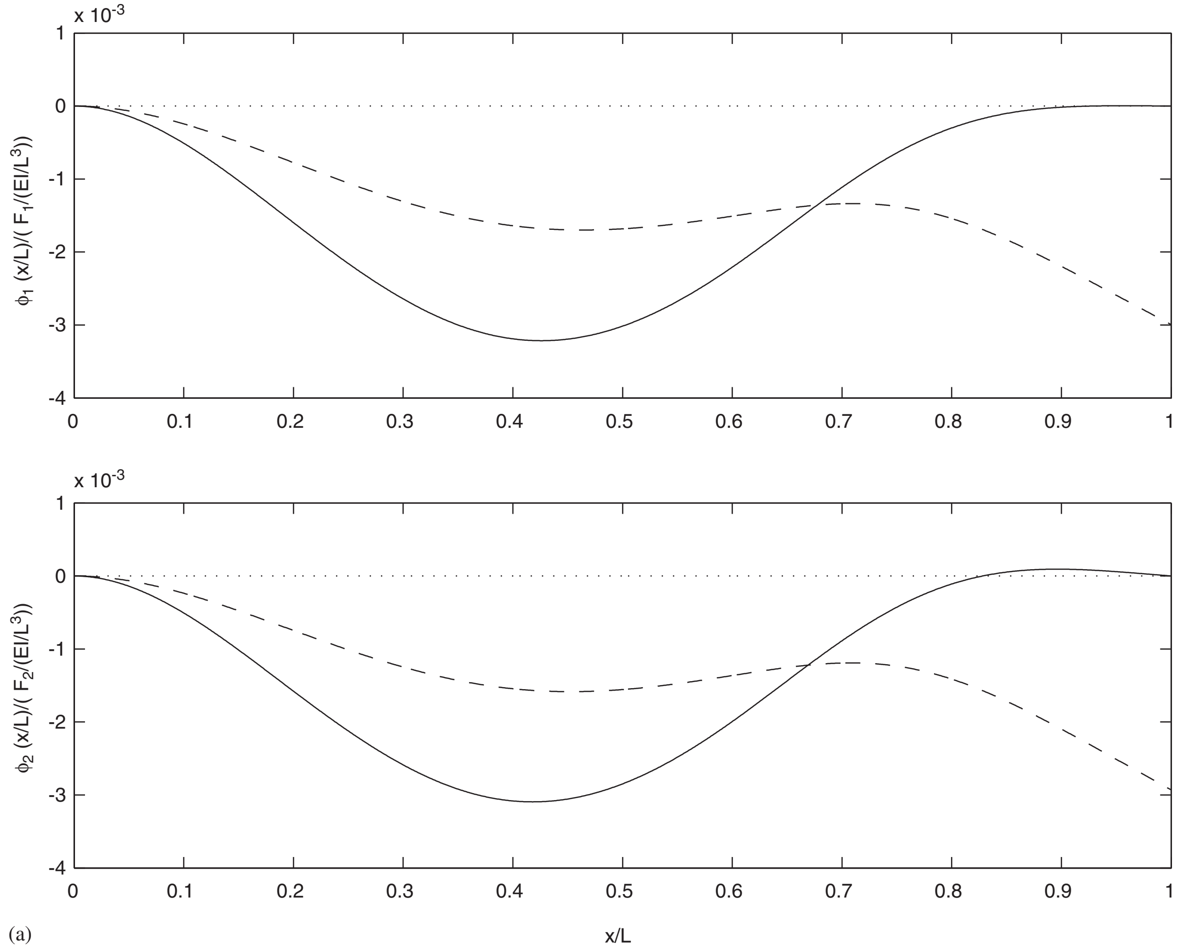
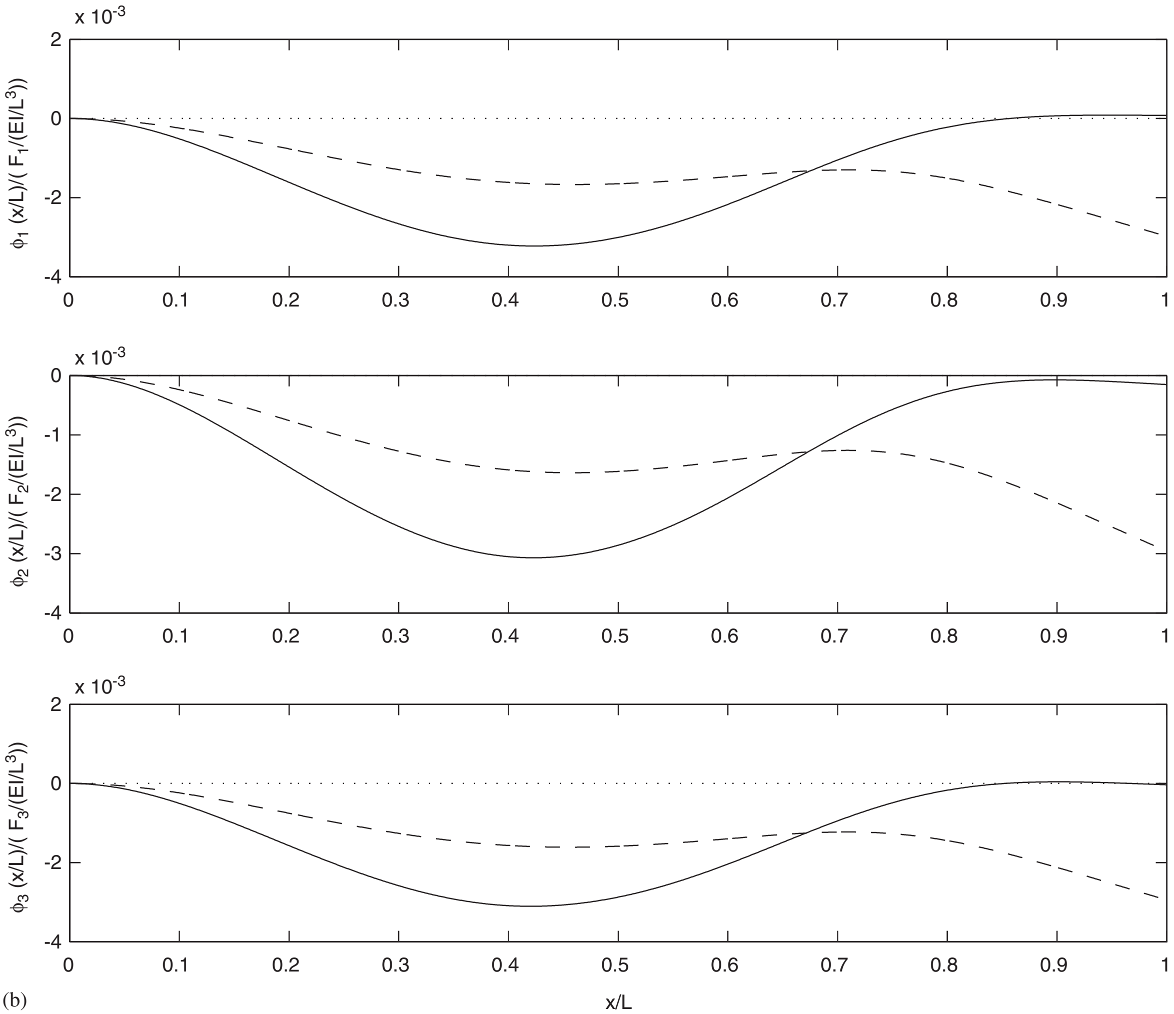


Fig. 4. The steady-state deformed shapes of a uniform fixed-free Euler–Bernoulli beam with (solid line) and without (dotted line) oscillator attachments. (a) The system parameters are $k_1 = 120EI/L^3$, $k_2 = 180EI/L^3$, $m_1 = 1.24870 \times 10^{-1}\rho L$ and $m_2 = 1.75781 \times 10^{-1}\rho L$. The attachment and node locations are collocated, $x_a = x_n = 1.00L$. The excitation frequencies are $\omega_1 = 31\sqrt{EI/(\rho L^4)}$, $\omega_2 = 32\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.76L$. (b) The system parameters, attachment, node and forcing locations are identical to those of (a). The new excitation frequencies are now $\omega_1^{new} = 31.25\sqrt{EI/(\rho L^4)}$, $\omega_2^{new} = 31.50\sqrt{EI/(\rho L^4)}$ and $\omega_3^{new} = 31.75\sqrt{EI/(\rho L^4)}$.



(b)

Fig. 4. (*Continued*)

stationary for excitation frequencies in the range of $\omega_1 \leqslant \omega \leqslant \omega_2$, because a slight perturbation in the excitation frequency often only leads to a slight perturbation in the system response. Consider a fixed-free beam subjected to a localized input at $x_f = 0.76L$, where $\omega_1 = 31\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 32\sqrt{EI/(\rho L^4)}$. For $x_a = x_n = 1.00L$, two oscillators are attached to the beam, with $k_1 = 120EI/L^3$, $k_2 = 180EI/L^3$, $m_1 = 1.24870 \times 10^{-1}\rho L$ and $m_2 = 1.75781 \times 10^{-1}\rho L$. The oscillator parameters are tuned to ω_1 and ω_2. Fig. 4(a) shows the steady-state deformed shape of the beam due to $F_r e^{j\omega_r t}$, and Fig. 4(b) illustrates the deformed shape of the beam due to a harmonic input with new frequencies of $\omega_1^{\text{new}} = 31.25\sqrt{EI/(\rho L^4)}$, $\omega_2^{\text{new}} = 31.50\sqrt{EI/(\rho L^4)}$, and

$\omega_3^{\text{new}} = 31.75\sqrt{EI/(\rho L^4)}$, respectively. Note that even though the excitation frequencies have changed, the absorbers, originally tuned to ω_1 and ω_2, still result in very small amplitudes for the beam at $x = 1.0L$ for frequencies between ω_1 and ω_2. This has practical benefits because it allows us to impose a point of nearly zero displacement for a harmonic input whose excitation frequency has the tendency to drift.

3.2. non-collocated case, one node

Consider a simply supported beam with a concentrated force applied at $x_f = 0.76L$. The localized force consists of two harmonics, with forcing frequencies of $\omega_1 = 33\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 49\sqrt{EI/(\rho L^4)}$. For a given application, it is desired to have a node at $x_n = 0.80L$. However, due to space constraint, oscillators cannot be attached at that location but at some other point, say $x_a = 0.23L$. In this case, Eq. (26) can be used to obtained the required mass parameters in order to induce a node, assuming that all the stiffness parameters are specified. For $k_1 = k_2 = 25EI/L^3$, solving Eq. (26) using *fsolve* gives $m_1 = 2.46948 \times 10^{-2}\rho L$ and $m_2 = 1.05129 \times 10^{-2}\rho L$. Fig. 5 illustrates the steady-state deformed shape of the beam. Note that a node is imposed at $0.80L$ as desired, and that for the given set of system parameters, the beam in the region between $0.80L < x < 1.00L$ experiences much smaller vibration compared to the beam with no oscillator attachments.

Consider a fixed-free beam with a concentrated force applied at $x_f = 0.90L$. The localized force consists of two harmonics with frequencies of $\omega_1 = 13\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 41\sqrt{EI/(\rho L^4)}$. A node is desired at $x_n = 1.00L$, and the attachment location is at $x_a = 0.75L$. The oscillator stiffnesses are $k_1 = k_2 = 18EI/L^3$, and the mass parameters are found by solving Eq. (26), which yields $m_1 = 9.09231 \times 10^{-2}\rho L$ and $m_2 = 1.05890 \times 10^{-2}\rho L$. Fig. 6 shows the steady-state shape of the deformed beam. Note that for the beam with oscillators, a node is induced at the tip, while for the beam without any attachments, its tip experiences large deflection.

3.3. Collocated, multiple nodes

Suppose one wishes to impose multiple, say q, nodes, for a linear structure that is subjected to an input consisting of p harmonics. This can be achieved by attaching q sets of p oscillators to the structure at distinct locations, as shown in Fig. 7. If the attachment and node locations are collocated, one can easily show that to induce nodes at $x_a^i = x_n^i$, for $i = 1, \ldots, q$, one simply tunes each oscillator natural frequency at x_a^i to a particular excitation frequency.

Consider a simply supported beam that is forced with an input at $x_f = 0.71L$, which consists of two harmonics with frequencies $\omega_1 = 13\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 41\sqrt{EI/(\rho L^4)}$. Two nodes are desired at $x_n^1 = 0.24L$ and $x_n^2 = 0.26L$, and the attachment and node locations are collocated. A total of four oscillators are attached to the beam, two at $x_a^1 = x_n^1$ and two at $x_a^2 = x_n^2$. For simplicity, let the oscillators all have stiffnesses $k_i = 18EI/L^3 = k$. Assume the oscillators with masses m_1 and m_2 are attached at x_a^1, and the absorbers with masses m_3 and m_4 are attached at x_a^2.

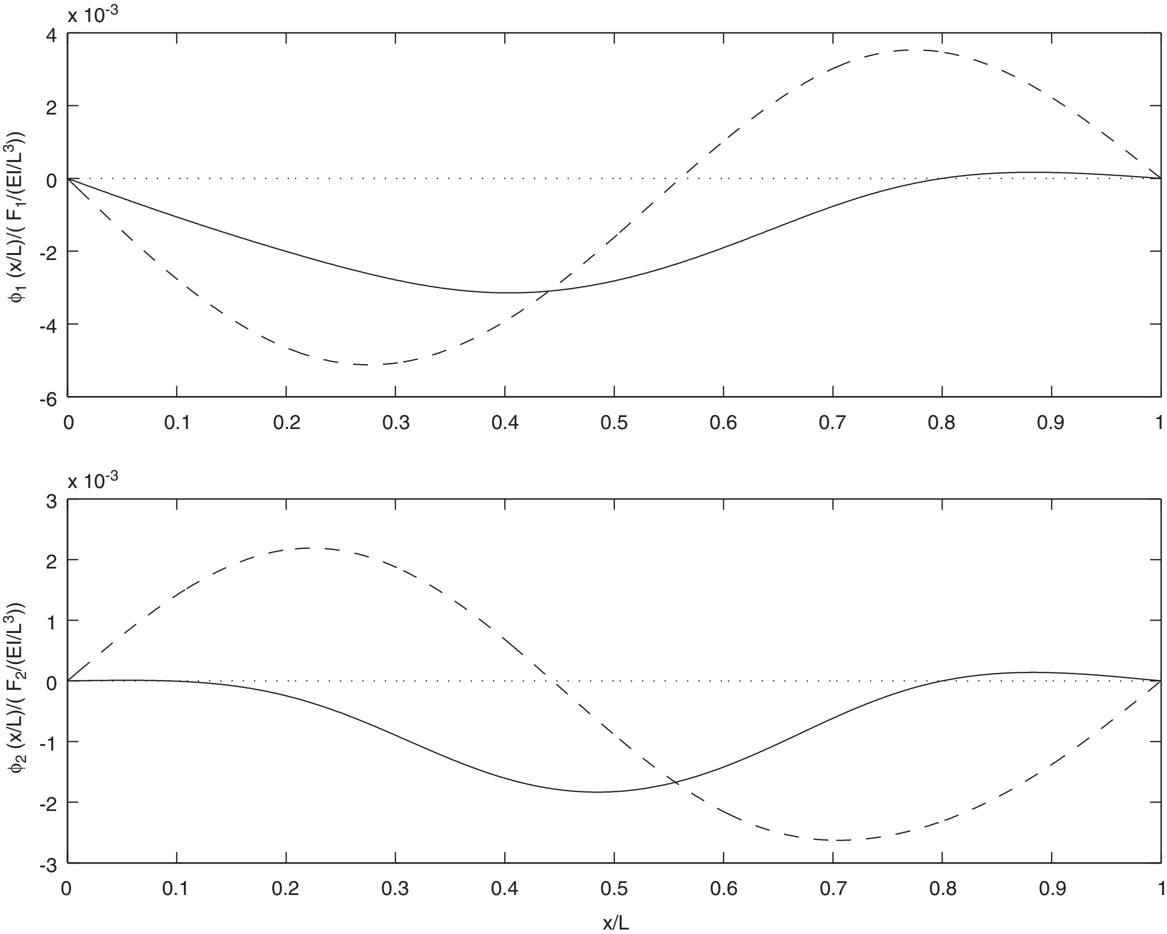


Fig. 5. The normalized steady-state deformed shape of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $k_1 = k_2 = 25EI/L^3$, $m_1 = 2.46948 \times 10^{-2}\rho L$ and $m_2 = 1.05129 \times 10^{-2}\rho L$. The attachment and node locations are non-collocated, $x_a = 0.23L$ and $x_n = 0.80L$. The excitation frequencies are $\omega_1 = 33\sqrt{EI/(\rho L^4)}$, $\omega_2 = 49\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.76L$.

The masses are selected such that $m_1 = m_3 = k/\omega_1^2$ and $m_2 = m_4 = k/\omega_2^2$. Physically, the first and third oscillators are tuned to impose nodes at x_a^1 and x_a^2 for the first excitation frequency, and the second and fourth are tuned to impose nodes at x_a^1 and x_a^2 for the second excitation frequency. Together, each set of oscillators will induce a node at the attachment location for ω_1 and ω_2. Fig. 8 shows the deformed shape of the beam for each harmonic. By imposing nodes that are closely spaced, it is possible to induce a region of nearly zero vibration. For this particular example, by enforcing nodes at $0.24L$ and $0.26L$, the vibration of the combined system is nearly quenched within the region between 0 and $0.3L$.

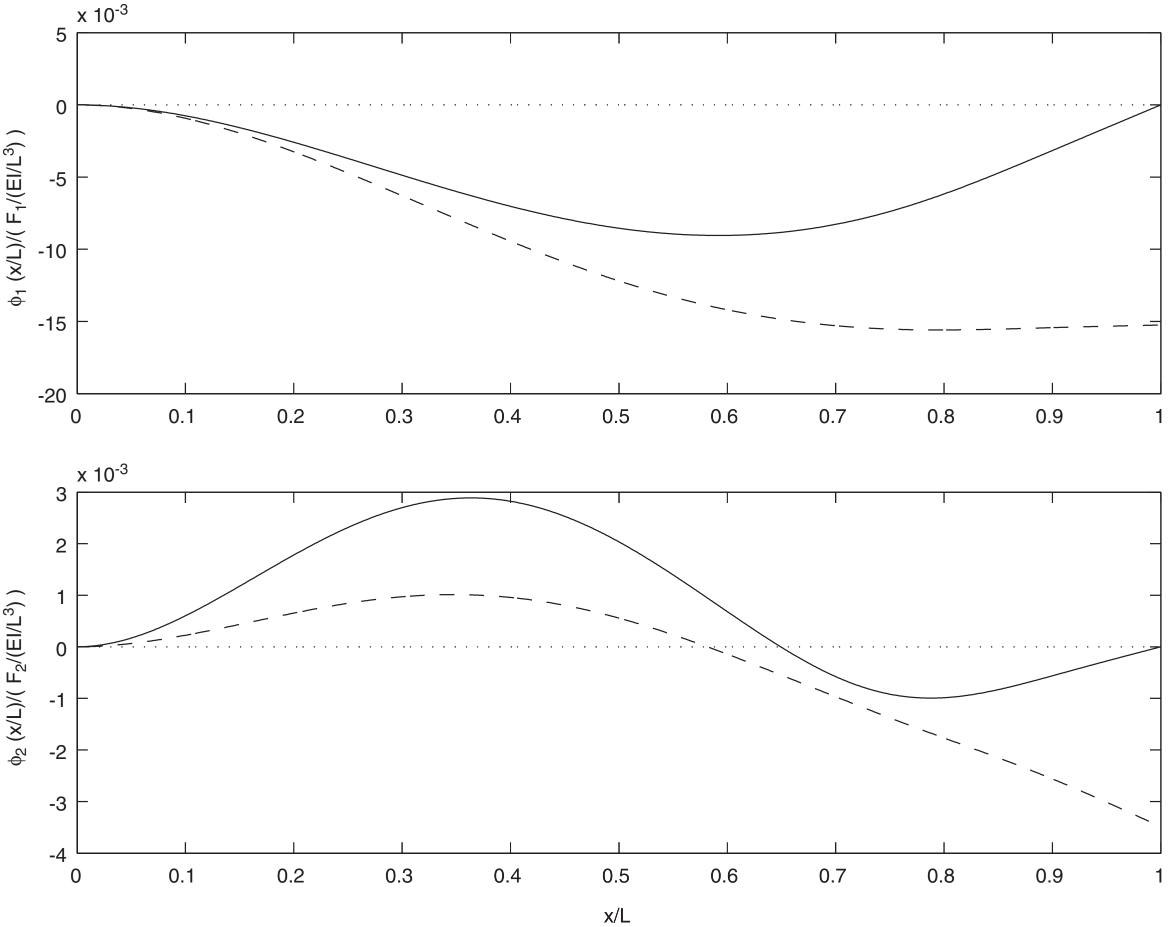


Fig. 6. The normalized steady-state deformed shape of a uniform fixed-free Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $k_1 = k_2 = 18EI/L^3$, $m_1 = 9.09231 \times 10^{-2}\rho L$ and $m_2 = 1.05890 \times 10^{-2}\rho L$. The attachment and node locations are non-collocated, $x_a = 0.75L$ and $x_n = 1.00L$. The excitation frequencies are $\omega_1 = 13\sqrt{EI/(\rho L^4)}$, $\omega_2 = 41\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.90L$.

3.4. Non-collocated, multiple nodes

Consider now the most interesting and challenging case of imposing q nodes at $(x_n^i)_{i=1,\dots,q}$, where the node and attachment locations, $(x_a^i)_{i=1,\dots,q}$, are not collocated, when the linear structure is subjected to p harmonics. To impose nodes at q locations for each harmonic, q absorbers are needed. Thus, a total of pq oscillators will be attached to the structure. Assume that the first p oscillators, with parameters $(m_i, k_i)_{i=1,\dots,p}$, are attached at x_a^1, and the next p oscillators, with parameters $(m_i, k_i)_{i=p+1,\dots,2p}$, are attached at x_a^2, etc. After some algebra, one can show that for this

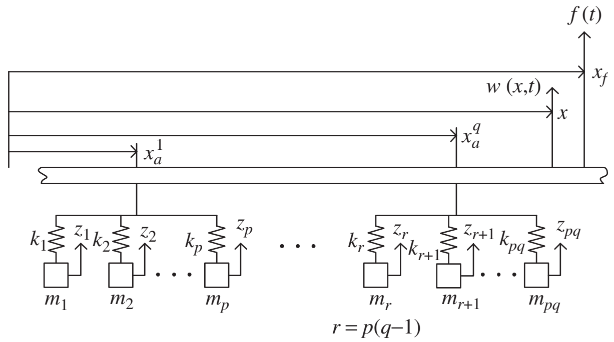


Fig. 7. An arbitrarily supported elastic structure that is subjected to a localized force and carrying multiple sets of sprung masses.

general case, Eq. (16) becomes

$$\left([K^d] + \sum\nolimits_{i=1}^{q} \alpha_i \mathbf{u}(x_a^i)\mathbf{u}^{\mathrm{T}}(x_a^i) - \omega^2[M^d]\right)\bar{\boldsymbol{\eta}} = F\mathbf{u}(x_f), \tag{37}$$

where

$$\alpha_r = \sum_{i=(r-1)p+1}^{rp} \frac{k_i m_i \omega^2}{m_i\omega^2 - k_i}, \quad r = 1\ldots,q. \tag{38}$$

To impose nodes at multiple locations, Eq. (20) becomes, not surprisingly, substantially more complicated as follows

$$\mathbf{u}^{\mathrm{T}}(x_n^r)\left([K^d] + \sum_{i=1}^{q} \alpha_i \mathbf{u}(x_a^i)\mathbf{u}^{\mathrm{T}}(x_a^i) - \omega^2[M^d]\right)^{-1} F\mathbf{u}(x_f) = 0, \quad r = 1\ldots,q. \tag{39}$$

Eq. (39) leads to a total of q equations for each excitation frequency. In order to induce q nodes when the input consists of p harmonics, a total of pq equations need to solved simultaneously for the desired masses, m_i, where $i = 1,\ldots,pq$.

Eq. (39) requires one to invert an $N \times N$ matrix. Because *fsolve* finds the solution to a system of nonlinear equations iteratively, for large values of N, solving Eq. (39) directly may be computationally taxing. Fortunately, the terms inside the parenthesis can be expressed as

$$[K^d] + \sum_{i=1}^{q} \alpha_i \mathbf{u}(x_a^i)\mathbf{u}^{\mathrm{T}}(x_a^i) - \omega^2[M^d] = [A] + [V][V]^{\mathrm{T}}, \tag{40}$$

where

$$[A] = [K^d] - \omega^2[M^d] \quad \text{and} \quad [V] = [U][D]. \tag{41}$$

Matrix $[U]$ is given by

$$[U] = [\,\mathbf{u}(x_a^1)\ \ \mathbf{u}(x_a^2)\ \ \ldots\ \ \mathbf{u}(x_a^i)\ \ \ldots\ \ \mathbf{u}(x_a^q)\,], \tag{42}$$

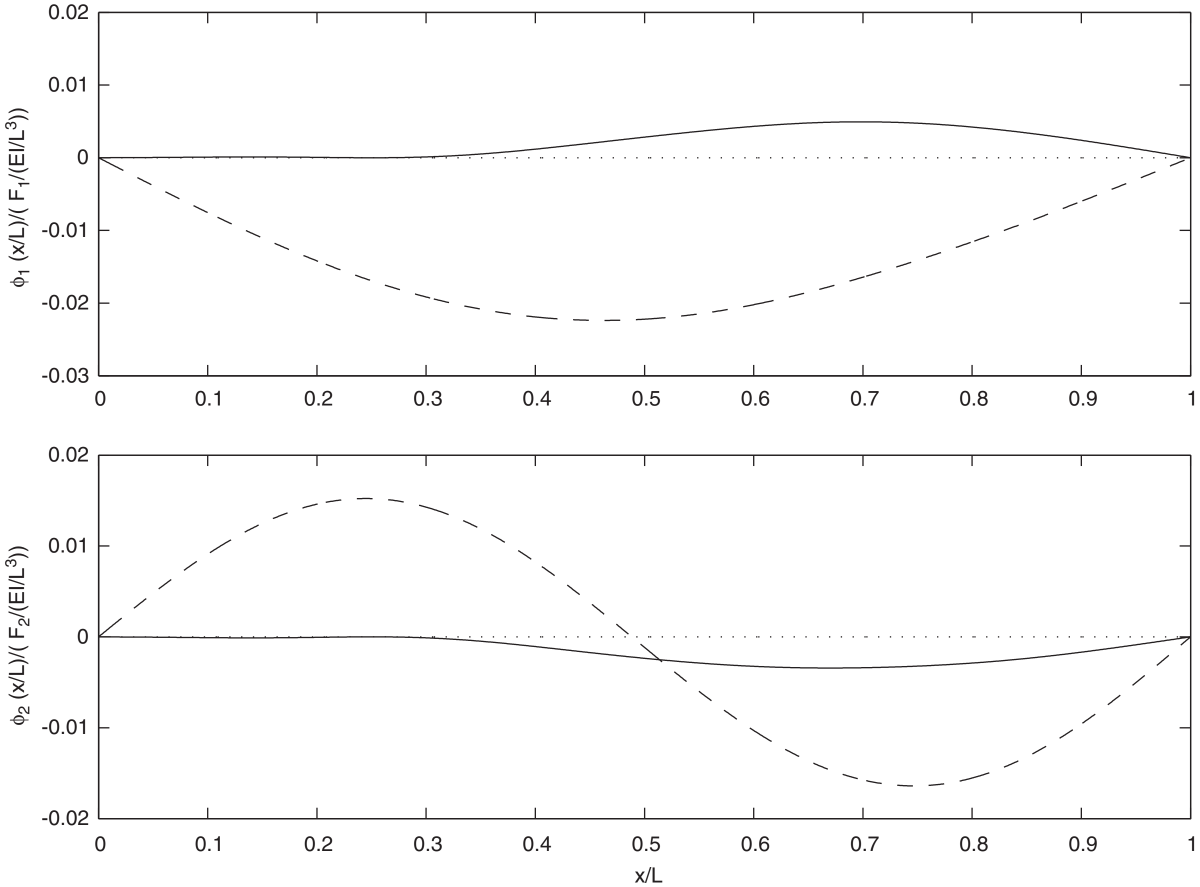


Fig. 8. The normalized steady-state deformed shape of a uniform simply supported Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $k_i = 18EI/L^3 = k$, $m_1 = m_3 = 1.06509 \times 10^{-1}\rho L$ and $m_2 = m_4 = 1.07079 \times 10^{-2}\rho L$. The attachment and node locations are collocated, $x_a^1 = x_n^1 = 0.24L$ and $x_a^2 = x_n^2 = 0.26L$. The excitation frequencies are $\omega_1 = 13\sqrt{EI/(\rho L^4)}$, $\omega_2 = 41\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.71L$.

and matrix $[D]$ is diagonal whose elements are the square roots of α_i,

$$[D] = \mathrm{diag}[\sqrt{\alpha_i}]. \tag{43}$$

Matrices $[U]$, $[D]$ and $[V]$ are of sizes $(N \times q)$, $(q \times q)$ and $(N \times q)$, respectively. Eq. (40) can be inverted using the Sherman–Morrison formula [18] as follows:

$$\left([A] + [V][V]^{\mathrm{T}}\right)^{-1} = [A]^{-1} - [A]^{-1}[V]\left([I] + [V]^{\mathrm{T}}[A]^{-1}[V]\right)^{-1}[V]^{\mathrm{T}}[A]^{-1}. \tag{44}$$

Because $[A]$ is diagonal, its inverse is trivial to obtain. Moreover, the resulting matrix inside the parenthesis is now of size $q \times q$. For $q << N$, it is clearly more efficient to invert the right-hand side than the left-hand side of Eq. (40). Thus, Eq. (39) is rewritten as

$$\mathbf{u}^{\mathrm{T}}(x_n^r)\Big\{[A]^{-1}-[A]^{-1}[V]\big([I]+[V]^{\mathrm{T}}[A]^{-1}[V]\big)^{-1}[V]^{\mathrm{T}}[A]^{-1}\Big\}F\mathbf{u}(x_f)=0,\quad r=1\ldots,q \tag{45}$$

and this equation will be used to solve for the required masses m_i in order to induce q nodes for p harmonics.

Consider a cantilever beam with a localized force applied at $x_f = 0.75L$. The external force consists of two harmonics with frequencies of $\omega_1 = 23\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 55\sqrt{EI/(\rho L^4)}$. Compared with the results of Table 1, note that the excitation frequencies are distinct from the natural frequencies of a fixed-free beam. Nodes are desired at $x_n^1 = 0.65L$ and $x_n^2 = 1.00L$. Because

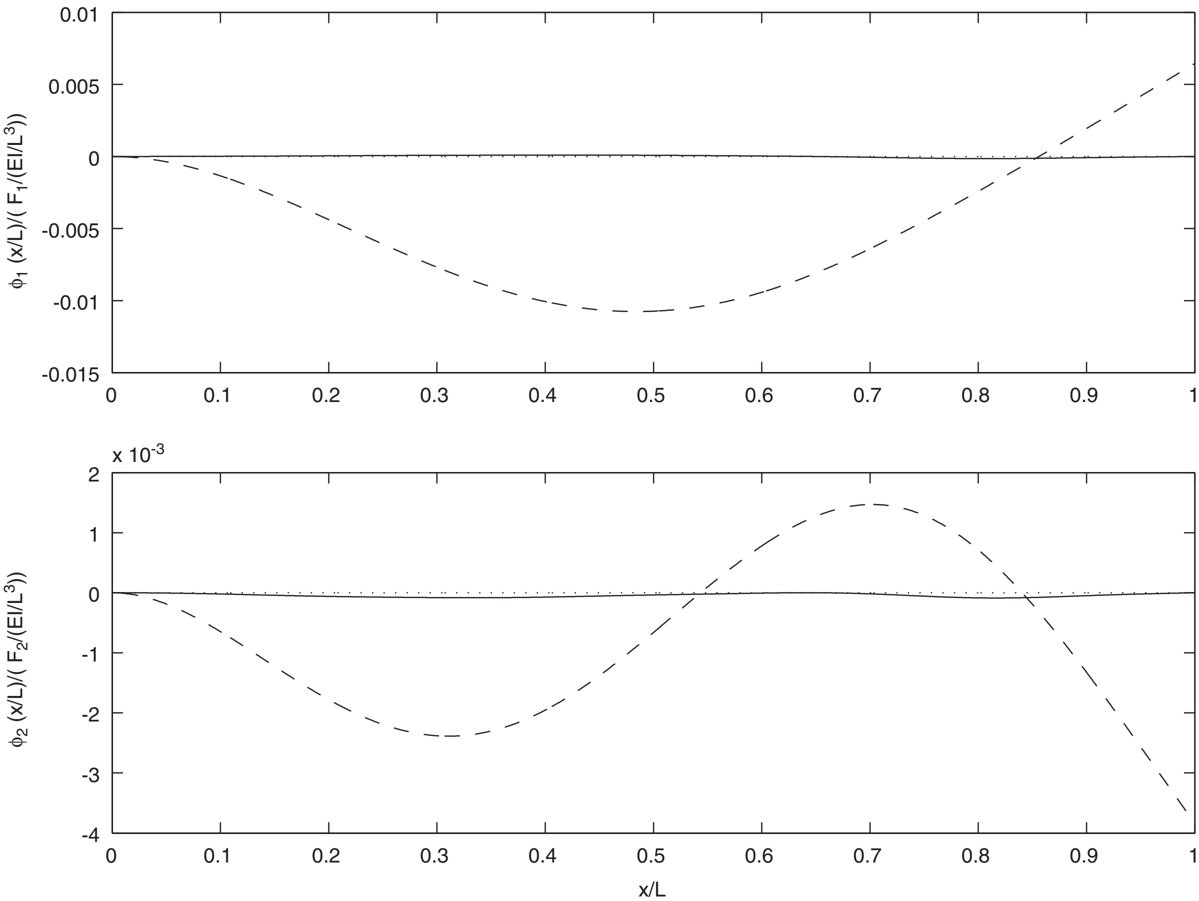


Fig. 9. The normalized steady-state deformed shape of a uniform fixed-free Euler–Bernoulli beam with (solid line) and without (dotted line) an oscillator attachment. The system parameters are $k_1 = k_3 = 20EI/L^3$, $k_2 = k_4 = 35EI/L^3$, $m_1 = 3.77780 \times 10^{-2}\rho L$, $m_2 = 1.15596 \times 10^{-2}\rho L$, $m_3 = 3.78452 \times 10^{-2}\rho L$ and $m_4 = 1.15849 \times 10^{-2}\rho L$. The attachment and node locations are non-collocated, $x_a^1 = 0.81L$, $x_a^2 = 0.83L$, $x_n^1 = 0.65L$ and $x_n^2 = 1.00L$. The excitation frequencies are $\omega_1 = 23\sqrt{EI/(\rho L^4)}$, $\omega_2 = 55\sqrt{EI/(\rho L^4)}$, and the forcing location is $x_f = 0.75L$.

of space constraint, two sets of oscillators, each consisting of two absorbers, are attached at $x_a^1 = 0.81L$ and $x_a^2 = 0.83L$. The stiffnesses are specified to be $k_1 = k_3 = 20EI/L^3$ and $k_2 = k_4 = 35EI/L^3$. Solving Eq. (45) simultaneously using *fsolve* yields $m_1 = 3.77780 \times 10^{-2}\rho L$, $m_2 = 1.15596 \times 10^{-2}\rho L$, $m_3 = 3.78452 \times 10^{-2}\rho L$ and $m_4 = 1.15849 \times 10^{-2}\rho L$. The normalized steady-state deformation of the beam, $\phi_r(x)/(F_r/(EI/L^3))$, due to $F_r e^{j\omega_r t}$ is shown in Fig. 9 Observe that nodes are induced at the desired locations at $0.65L$ and $1.00L$ nodes. Moreover, note that the vibration level of the entire beam is dramatically reduced compared to the beam without any attachments. Thus, by attaching properly tuned oscillators to the cantilever beam, one can induce nodes at the desired locations and substantially quench the vibration of the entire structure without using any rigid supports.

A finite element model is constructed to validate the aforementioned example. The beam is discretized into 100 equal beam elements, and sprung masses are attached at x_a^1 and x_a^2. The stiffnesses of the absorbers are identical to those listed above, and the mass parameters correspond to those obtained by solving Eq. (45). Table 6 compares the natural frequencies of the combined system, obtained by using the assumed modes and the finite element method. Note the excellent agreement between the two sets of results. Interestingly, the third and sixth natural frequencies of the combined system are close but not identical to the excitation frequencies. Table 7 shows the amplitudes of the absorber masses. Again, note how well the results track one another.

Finally, a few words about determining the mass parameters for the non-collocated case are warranted. The MATLAB routine *fsolve* is used to obtain the desired mass parameters in order to induce a node at the specified location. In application, *fsolve* requires a set of initial guesses of the unknown parameters to be provided. Numerical experiments showed that when the attachment and node locations are non-collocated, one can use the mass parameters for the collocated case as initial guesses, and in all the examples consider, *fsolve* converges to a set of theoretically feasible solution. Not surprisingly, as the number of harmonics, p, and node locations, q, increases, the problem becomes more computationally intensive because a total of pq nonlinear algebraic equations must be satisfied simultaneously.

Using the assumed modes method, a simple and efficient approach has been developed to solve the inverse problem of imposing one or multiple nodes anywhere along an arbitrarily supported

Table 6
The first six natural frequencies of a uniform cantilever Euler-Bernoulli beam carrying four undamped oscillators, of stiffnesses $k_1 = k_3 = 20EI/L^3$, $k_2 = k_4 = 35EI/L^3$, and masses $m_1 = 3.77780 \times 10^{-2}\rho L$, $m_2 = 1.15596 \times 10^{-2}\rho L$, $m_3 = 3.78452 \times 10^{-2}\rho L$ and $m_4 = 1.15849 \times 10^{-2}\rho L$

Natural frequency	Assumed modes ($N = 10$)	Finite element ($N = 100$)
ω_1'	3.17345E+00	3.17345E+00
ω_2'	2.16855E+01	2.16854E+01
ω_3'	2.30404E+01	2.30401E+01
ω_4'	2.50124E+01	2.50113E+01
ω_5'	5.48986E+01	5.48972E+01
ω_6'	5.53695E+02	5.53651E+02

The first two oscillators are attached at $x_a^1 = 0.81L$, and the last two oscillators are attached at $x_a^2 = 0.83L$. The natural frequencies, ω_i', of the combined system are non-dimensionalized by dividing by $\sqrt{EI/(\rho L^4)}$.

Table 7
The mass amplitudes for the beam and oscillators of Table 6

Mass amplitudes	Assumed modes ($N = 10$)	Finite element ($N = 100$)
z_1 (for ω_1)	−1.89639E–01	−1.91073E–01
z_2 (for ω_1)	−1.77139E–04	−1.78479E–04
z_3 (for ω_1)	1.40241E–01	1.41359E–01
z_4 (for ω_1)	−1.71029E–04	−1.72393E–04
z_1 (for ω_2)	1.83714E–05	1.80793E–05
z_2 (for ω_2)	−9.40288E–02	−9.25335E–02
z_3 (for ω_2)	1.80946E–05	1.78757E–05
z_4 (for ω_2)	6.77194E–02	6.69000E–02

The beam is subjected to a localized force at $x_f = 0.75L$ that consists of two harmonics with frequencies $\omega_1 = 23\sqrt{EI/(\rho L^4)}$ and $\omega_2 = 55\sqrt{EI/(\rho L^4)}$. The mass amplitudes are normalized by dividing by $F_r/(EI/L^3)$, where F_r denotes the forcing amplitude of the rth harmonic.

elastic structure that is subjected to a localized force consisting of multiple harmonic excitations. This has practical benefits because it allows any point along the structure to remain stationary without using any rigid supports, and it enables certain regions of the structure to undergo very small deflections, thereby suppressing vibration in those sections. Finally, the mass parameters obtained by using *fsolve* are generally not unique. Thus, multiple sets of theoretically feasible solutions are possible. In application, another important design specification is governed by the vibration of the absorber masses. If the vibration amplitudes of these masses are large, then the stiffnesses can be increased to reduce the mass displacements. If the deflections of the absorber masses are still too large, then theoretically feasible solutions would not be practical, and it would be necessary to introduce dampers to the vibration absorbers. This interesting problem of imposing the additional constraint of maximum vibration amplitude of the masses will be pursued in a future research project.

4. Conclusions

Oscillators can be used to impose one or multiple nodes anywhere along an elastic structure that is subjected to a localized force consisting of multiple harmonic excitations. When the parameters of the sprung masses are properly chosen, one or more nodes can always be induced at the attachment locations for an input with multiple harmonics and for any excitation location. When the attachment and the node locations are not collocated, it is only possible to induce nodes at certain locations along the structure. In addition, if a node is specified for two excitation frequencies ω_1 and ω_2 that are closely spaced, then that particular location remains nearly stationary for excitation frequencies between ω_1 and ω_2. Moreover, if the node locations are properly selected, a region of nearly zero amplitudes can be imposed along the elastic structure for a given localized harmonic force without using any rigid supports, effectively quenching vibration in that segment of the structure. A detailed procedure to assist in the selection of the attached spring–mass systems was outlined, and numerical experiments were performed to validate the utility of the proposed scheme of imposing a single or multiple nodes during harmonic excitations for the collocated and non-collocated cases.

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